If a number is divisible by 3, then so is the sum of its digits

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1 Introduction

There was a trick we learned in elementary school: if the sum of the digits of a number is divisible by 3, then the number itself is divisble by 3.

Example 1. *Is 54,321 divisible by 3? The sum of digits* $5+4+3+2+1=15$, *which is divisible by 3, so 54,321 should be divisible by 3 according to this rule, and lo,* $54,321 = 3 \cdot 18,107$.

Is that true for all numbers?

2 Proving it

We write numbers as strings of decimal digits like so:

$$\begin{aligned} 54,321 &= 5 \cdot 10,000 + 4 \cdot 1,000 + 3 \cdot 100 + 2 \cdot 10 + 1 \\ &= 5 \cdot 10^4 + 4 \cdot 10^3 + 3 \cdot 10^2 + 2 \cdot 10^1 + 1 \cdot 10^0 \end{aligned}$$

More precisely, we write an integer D as a string of decimal digits: $d_{n-1}, \ldots, d_1, d_0$, which represents the equation:

$$\begin{aligned} D &= d_{n-1}10^{n-1} + \cdots + d_1 10^1 + d_0 10^0 \\ &= \sum_{i=0}^{n-1} d_i 10^i \end{aligned}$$

Given that definition of the digits of a number, we can prove the theorem. First, a couple of mini-theorems:

Lemma 1. *For all integer polynomials, $(x+1)^n = xk_n + 1$ for some integer k_n. In other words, $(x+1)^n - 1$ is divisible by x.*

Proof. By induction. First the base case where $n = 0$: k_0 is trivially 0.

$$\begin{aligned}(x+1)^0 &= 1 \\ &= x \cdot 0 + 1\end{aligned}$$

Now, the inductive step. Assuming n, prove $n+1$:

$$\begin{aligned}(x+1)^{n+1} &= (x+1)(x+1)^n \\ &= (x+1)(xk_n + 1) && \text{assuming } n\text{: } (x+1)^n = xk_n + 1 \\ &= xxk_n + x + xk_n + 1 \\ &= x(xk_n + k_n + 1) + 1 \\ &= x(k_{n+1}) + 1 && \text{where } k_{n+1} = xk_n + k_n + 1\end{aligned}$$

We've proven the $n+1$ case: $(x+1)^{n+1} = xk_{n+1} + 1$, and by induction this is be true for all $n \geq 0$. □

Lemma 2. *if $D = pk + q$, then D is divisible by p if and only if q is divisible by p.*

Proof. Assume q is divisible by p. Then, $q = pj$ for some integer j, and

$$\begin{aligned}D &= pk + q \\ &= pk + pj \\ &= p(k+j)\end{aligned}$$

thus D is divisible by p.

Now, assume q is **not** divisible by p. Then, $q = pj + r$ for some $0 < r < p$, and

$$\begin{aligned}D &= pk + q \\ &= pk + pj + r \\ &= p(k+j) + r\end{aligned}$$

and since $0 < r < p$, D is **not** divisible by p.

Thus $D = pk + q$ is divisible by p if and only if q is divisible by p. □

Now we can get back to the main theorem.

Theorem 1. *If an integer D is written as a string of digits $d_{n-1}, \ldots, d_1, d_0$ where $D = \sum_{i=0}^{n-1} d_i 10^i$, then D is divisible by 3 if and only if the sum of its digits $S = \sum_{i=0}^{n-1} d_i$ is divisible by 3.*

Proof. The proof uses the simple fact that $10 = (9 + 1)$:

$$\begin{aligned}
D &= \sum_{i=0}^{n-1} d_i 10^i \\
&= \sum_{i=0}^{n-1} d_i (9+1)^i \\
&= \sum_{i=0}^{n-1} d_i (9k_i + 1) && \text{by Lemma 1} \\
&= 9 \sum_{i=0}^{n-1} d_i k_i + \sum_{i=0}^{n-1} d_i \\
&= 9k + S && \text{where } S \text{ is the sum of the digits of } D
\end{aligned}$$

So $D = 9k + S$, and by Lemma 2, D is divisible by 9 if and only if the sum of its digits, $S = \sum_{i=0}^{n-1} d_i$ is also divisible by 9. That's an interesting result, but we were trying to prove that statement for 3. However, since $9 = 3 \cdot 3$:

$$\begin{aligned}
D &= 9k + S \\
&= 3 \cdot 3k + S \\
&= 3j + S
\end{aligned}$$

Lemma 2 works again to prove that D is divisible by 3 if and only if the sum of its digits, $S = \sum_{i=0}^{n-1} d_i$ is also divisible by 3.

□

3 Epilogue

This proof used the fact that we write integers in base 10, and $10 = (9+1)$, and thus if the sum of a number's digits in base 10 is divisible by 9 or 3, then so is the number itself. This works for other bases too. For example, if the number's digits are in base 8, this rule will work for all divisors of $8 - 1 = 7$. For example, $5432_8 = 2842_{10} = 7 \cdot 406_{10}$, and $5 + 4 + 3 + 2 = 14_{10}$✓